Variance Ratio Testing for Fractional Cointegration with Non-Linear Smooth Break Approximations using Fourier Functions

Chingnun Lee†

†Institute of Economics, National Sun Yat-sen University

Department of Economics, Soochow University, June 2025

Outline

Traditional Parametric Framework

Unit Root

- Augmented Dickey-Fuller Test (Dickey and Fuller, 1979)
- Phillips and Perron (1988)

Cointegration

- Engle and Granger (1987)
- Johansen Test (Johansen, 1988, 1991)

Augmented Dickey-Fuller Test

Dickey and Fuller (1979)

An $AR(p)$ process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t, ^1 \tag{1}$$

where ε_t is an i.i.d. sequence with mean zero, variance σ^2, and finite fourth moment.

[1] The model can include an intercept and/or a time trend.

Augmented Dickey-Fuller Test

Dickey and Fuller (1979)

The equation (1) can be rewriten as

$$Y_t = \rho Y_{t-1} + \xi_1 \Delta Y_{t-1} + \xi_2 \Delta Y_{t-2} + \cdots + \xi_{p-1} \Delta Y_{t-p+1} + \varepsilon_t, \qquad (2)$$

where

$$\begin{aligned} \rho &= \phi_1 + \phi_2 + \cdots + \phi_p, \\ \xi_j &= -\left[\phi_{j+1} + \phi_{j+2} + \cdots + \phi_p\right], \end{aligned}$$

for $j = 1, 2, \ldots, p-1$. Using OLS to estimate ρ, $\xi_1, \ldots, \xi_{p-1}$ yields $\hat{\rho}$, $\hat{\xi}_1, \ldots, \hat{\xi}_{p-1}$.

Augmented Dickey-Fuller Test

Dickey and Fuller (1979)

Theorem (Dickey and Fuller, 1979)

As $T \to \infty$, under $H_0 : \rho = 1$, we have

$$\frac{T(\hat{\rho} - 1)}{1 - \hat{\xi}_1 - \cdots - \hat{\xi}_{p-1}} \Rightarrow \frac{1}{2} \left(\int_0^1 W(r)^2 dr \right)^{-1} \left(W(1)^2 - 1 \right), \tag{3}$$

where T is the sample size and $W(r)$ is a standard Brownian motion.

Phillips-Perron Test

Phillips and Perron (1988)

An $AR(1)$ process with serially correlated errors

$$Y_t = \rho Y_{t-1} + u_t, \tag{4}$$

where $u_t = \sum_{j=0}^{\infty} \psi_i \varepsilon_{t-j}$, $\sum_{j=0}^{\infty} j|\psi_j| < \infty$, and ε_t is an i.i.d. sequence with mean zero, variance σ^2, and finite fourth moment.

Phillips-Perron Test

Phillips and Perron (1988)

Definition (Autocovariance)

$$\gamma_j \equiv E\left(u_t u_{t-j}\right), \text{ for } j = 0, 1, 2, \ldots \ldots \tag{5}$$

Definition (Long-run variance)

$$\lambda^2 \equiv \lim_{T\to\infty} E\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} u_t\right)^2 = \gamma_0 + 2\sum_{j=1}^{\infty}\gamma_j. \tag{6}$$

Phillips-Perron Test

Phillips and Perron (1988)

Let $\hat{\rho}$ be the OLS estimate based on equation (4). If the true value $\rho = 1$, then, as $T \to \infty$, we have

$$T\left(\hat{\rho}-1\right) \Rightarrow \frac{1}{2}\left(\int_0^1 W(r)^2 dr\right)^{-1}\left(W(1)^2 - \frac{\gamma_0}{\lambda^2}\right). \tag{7}$$

We need to construct modified statistics whose limiting distributions are independent of γ_0 and λ^2.

Phillips-Perron Test

Phillips and Perron (1988)

Newey-West estimator

$$\hat{\lambda}^2 = \hat{\gamma}_0 + 2\sum_{j=1}^{q} w_{jq}\hat{\gamma}_j, \tag{8}$$

$$\hat{\gamma}_j = \frac{1}{T}\sum_{t=j+1}^{T} \hat{u}_t\hat{u}_{t-j}, \tag{9}$$

where $w_{jq} = 1 - j/(q+1)$, $\hat{u}_t = Y_t - \hat{\rho}Y_{t-1}$, and q is the lag truncation parameter.

Phillips-Perron Test

Phillips and Perron (1988)

Theorem (Phillips and Perron, 1988)

Under $H_0 : \rho = 1$, *if* $q \to \infty$ *As* $T \to \infty$ *but* $q/T \to 0$, *then*

$$
\begin{aligned}
Z_\rho &\equiv T(\hat{\rho} - 1) - \frac{1/2(\hat{\lambda}^2 - \hat{\gamma}_0)}{\sum_{t=2}^{T} Y_{t-1}^2 / T^2} \\
&\Rightarrow \frac{1}{2} \left(\int_0^1 W(r)^2 dr \right)^{-1} \left(W(1)^2 - 1 \right), \qquad (10)
\end{aligned}
$$

where $W(r)$ *is a standard Brownian motion.*

Integrated Processes

Definition (An $I(d)$ Process)

The n-vector time series $\boldsymbol{Y}_t$ is said to be integrated of order d, denoted $\boldsymbol{Y}_t \in I(d)$, if

$$(1 - L)^d \boldsymbol{Y}_t = \boldsymbol{u}_t, \ t = 1, 2, \ldots, \tag{11}$$

where $\boldsymbol{u}_t$ is stationary process.

Example

1. A stationary process is an $I(0)$ process.
2. A unit root process is an $I(1)$ process.

Cointegration

Definition

The n-vector time series $\boldsymbol{Y}_t$ is cointegrated if

1. $\boldsymbol{Y}_t \in I(d)$;
2. There exists a nonzero $(n \times 1)$ vector $\mathbf{a}$ such that $\mathbf{a}'\boldsymbol{Y}_t \in I(d-b)$ for $b > 0$.

Remark

1. The vector $\mathbf{a}$ is called a *cointegrating vector*.
2. There may be $h < n$ linearly independent $(n \times 1)$ vectors $(\mathbf{a}_1, \ldots, \mathbf{a}_h)$ such that $\mathbf{A}'\boldsymbol{Y}_t \in I(d-b)$ is a $(h \times 1)$ vector, where $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_h)$ is a $(n \times h)$ matrix.

Resudual-Based Test for Cointegration

Engel and Granger (1987)

Consider the regression

$$Y_t = \mathbf{x}_t'\boldsymbol{\beta} + e_t. \tag{12}$$

We can estimate equation (12) by OLS, yielding the residual $\hat{e}_t$. The residual $\hat{e}_t$ can then be regressed on its own lagged value $\hat{e}_{t-1}$ without a constant term:

$$\hat{e}_t = \rho\hat{e}_{t-1} + u_t,\ t = 2, 3, \ldots, T, \tag{13}$$

yielding the estimate

$$\hat{\rho} = \sum_{t=2}^{T} \hat{e}_{t-1}\hat{e}_t \Big/ \sum_{t=2}^{T} \hat{e}_{t-1}^2. \tag{14}$$

Resudual-Based Test for Cointegration

Engel and Granger (1987)

Define the second-stage residuals

$$\hat{u}_t = \hat{e}_t - \hat{\rho}\hat{e}_{t-1},\ t = 2, 3, \ldots, T. \tag{15}$$

The Phillips-Perron statistics can be written as

$$Z_\rho = T(\hat{\rho} - 1) - \frac{1/2(\hat{\lambda}^2 - \hat{\gamma}_0)}{\sum_{t=2}^{T} \hat{e}_{t-1}^2 / T^2}. \tag{16}$$

Resudual-Based Test for Cointegration

Engel and Granger (1987)

Assumption (No Cointegration)

Consider an $(n \times 1)$ vector $\boldsymbol{Y}_t = (Y_t, \mathbf{x}_t')'$ such that

$$\Delta \boldsymbol{Y}_t = \sum_{j=0}^{\infty} \boldsymbol{\Psi}_j \boldsymbol{\varepsilon}_{t-j}, \tag{17}$$

for $\boldsymbol{\varepsilon}_t$ an i.i.d. vector with mean zero, variance $E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \mathbf{P}\mathbf{P}'$, and finite fourth moments, and where $\{j \cdot \boldsymbol{\Psi}_j\}_{j=0}^{\infty}$ is absolutely summable. Let $\boldsymbol{\Lambda} = (\boldsymbol{\Psi}_0 + \boldsymbol{\Psi}_1 + \cdots)\mathbf{P}$. Suppose that the $(n \times n)$ matrix $\boldsymbol{\Lambda}\boldsymbol{\Lambda}'$ is nonsingular.

Resudual-Based Test for Cointegration

Engel and Granger (1987)

Theorem (Phillips and Ouliaris, 1990)

Under Assumption, if $q \to \infty$ as $T \to \infty$ but $q/T \to 0$, then

$$Z_\rho \Rightarrow \int_0^1 Q(r)dQ(r) \Big/ \int_0^1 Q(r)^2 dr, \tag{18}$$

where

$Q(r) = W_1(r) - \int_0^1 W_1(r)\boldsymbol{W}_2(r)'dr \left[\int_0^1 \boldsymbol{W}_2(r)\boldsymbol{W}_2(r)'dr\right]^{-1} \boldsymbol{W}_2(r),$

Here, $(W_1(r), \boldsymbol{W}_2(r)')'$ denotes n-dimensional standard Brownian motion.

Cointegrated System

Consider the following $VAR(p)$ model, which we write in the VEC form:

$$\Delta \boldsymbol{Y}_t = \boldsymbol{\Pi} \boldsymbol{Y}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \boldsymbol{Y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{p-1} \Delta \boldsymbol{Y}_{t-1+1} + \boldsymbol{\varepsilon}_t, \tag{19}$$

where $E(\boldsymbol{\varepsilon}_t) = \mathbf{0}$, $E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \boldsymbol{\Omega}$ and $E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_s') = \mathbf{0}$ for $t \neq s$.

Cointegrated System

Theorem (Granger representation theorem)

Suppose that there are exactly h cointegrating relations among the elements of $\boldsymbol{Y}_t$, and the process can be represented as the pth-order VAR, then

1. *There exists an $(h \times n)$ matrix $\mathbf{A}'$ whose rows are linearly independent such that $\mathbf{A}'\boldsymbol{Y}_t$ is stationary.*
2. *There exists an $(n \times h)$ matrix $\mathbf{B}$ such that $\mathbf{\Pi} = \mathbf{B}\mathbf{A}'$.*

Likelihood Ratio Test

Johansen (1988)

The log likelihood of $(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_T)$ conditional on $(\boldsymbol{Y}_{-p+1}, \boldsymbol{Y}_{-p+2}, \ldots, \boldsymbol{Y}_0)$ is given by

$$\begin{aligned} &\mathcal{L}\left(\boldsymbol{\Omega}, \boldsymbol{\Gamma}_1, \ldots, \boldsymbol{\Gamma}_{p-1}, \boldsymbol{\Pi}\right) \\ = \; & -\frac{Tn}{2}\log(2\pi) - \frac{T}{2}\log|\boldsymbol{\Omega}| - \frac{1}{2}\sum_{t=1}^{T} \boldsymbol{\varepsilon}_t' \boldsymbol{\Omega}^{-1} \boldsymbol{\varepsilon}_t, \end{aligned} \tag{20}$$

where

$$\boldsymbol{\varepsilon}_t = \Delta \boldsymbol{Y}_t - \boldsymbol{\Pi}\boldsymbol{Y}_{t-1} - \boldsymbol{\Gamma}_1 \Delta \boldsymbol{Y}_{t-1} - \cdots - \boldsymbol{\Gamma}_{p-1}\Delta \boldsymbol{Y}_{t-p+1}. \tag{21}$$

Likelihood Ratio Test

Johansen (1988)

The likelihood ratio (LR) test statistic for the hypothesis that there are at most h cointegration vectors is given by

$$LR = 2\left(\mathcal{L}_A^* - \mathcal{L}_h^*\right), \tag{22}$$

where $\mathcal{L}_h^*$ represents the maximum value of the log-likelihood function under the restriction that there are exactly h cointegrating relations among the elements of $\boldsymbol{Y}_t$, and $\mathcal{L}_A^*$ denotes the maximum value of the log-likelihood function without any restrictions.

Likelihood Ratio Test

Johansen (1988)

Theorem (Johansen, 1988)

Under the hypothesis that there are h cointegrating vectors, then

$$LR \Rightarrow \operatorname{tr}(\mathbf{Q}), \tag{23}$$

where

$$\mathbf{Q} \equiv \left[\int_0^1 \boldsymbol{W}(r)d\boldsymbol{W}(r)'\right]' \left[\int_0^1 \boldsymbol{W}(r)d\boldsymbol{W}(r)'\right] \left[\int_0^1 \boldsymbol{W}(r)d\boldsymbol{W}(r)'\right],$$

and $\boldsymbol{W}(r)$ is $(n-h)$-dimensional standard Brownian motion.

Defects of the Parametric Model

Using the parametric model requires the following:

1. Specification of the short-run dynamics, such as an $AR(p)$ model.
2. Estimation of nuisance parameters, such as the long-run variance.

Breitung (2002) proposed a nonparametric ratio test for unit root and cointegration rank that does not require the above.

Variance Ratio statistic for Unit Root

Breitung (2002)

Under the null hypothesis that $Z_t \in I(1)$, Z_t satisfy the functional central limit theorem

$$T^{-1/2} Z_{[Tr]} \Rightarrow \sigma W(r), \tag{24}$$

where $0 < r \leq 1$, $W(r)$ is a standard Brownian motion and $\sigma^2 = \lim E(T^{-1/2} Z_T)^2$ is the long-run variance.

It is obvious to see that (24) has the nuisance parameter σ need to be eliminate.

Variance Ratio statistic for Unit Root

Breitung (2002)

Following Kwiatkowski *et al.* (1992), consider the following test statistic

$$\rho_B = T^{-2} \sum_{t=1}^{T} \hat{S}_t^2 \left[T^{-1} \sum_{t=1}^{T} \hat{Z}_t \right]^{-1}, \tag{25}$$

where $\hat{Z}_t = Y_t - \hat{\alpha}' \boldsymbol{D}_t$, $\hat{\alpha} = (\sum_{t=1}^{T} \boldsymbol{D}_t \boldsymbol{D}_t')^{-1} \sum_{t=1}^{T} \boldsymbol{D}_t Y_t$, $\hat{S}_t = \hat{Z}_1 + \hat{Z}_2 + \cdots + \hat{Z}_t$, Y_t is the observed time series and $\boldsymbol{D}_t$ is the deterministic term.

Variance Ratio statistic for Unit Root

Breitung (2002)

Theorem (Breitung, 2002)

Under the null hypothesis, we have

$$\begin{aligned}\rho_B &\Rightarrow \int_0^1 \left[\int_0^a \sigma W_j(r)dr\right]^2 da \left[\int_0^1 \sigma^2 W_j(r)^2 dr\right]^{-1} \\ &= \int_0^1 \left[\int_0^a W_j(r)dr\right]^2 da \left[\int_0^1 W_j(r)^2 dr\right]^{-1}, \qquad (26)\end{aligned}$$

where

$$W_0(r) \equiv W(r), \text{ for } \boldsymbol{D}_t = 0,$$

Variance Ratio statistic for Unit Root

Breitung (2002)

$$
\begin{aligned}
W_1(r) &\equiv W(r) - \int_0^1 W(a)da, \text{ for } \boldsymbol{D}_t = 1, \\
W_2(r) &\equiv W(r) - (4 - 6r)\int_0^1 W(a)da - (12r - 6) \\
&\quad \times \int_0^1 aW(a)da, \text{ for } \boldsymbol{D}_t = (1, t)'.
\end{aligned}
$$

It is easy to seen that the nuisance parameter σ is eliminated from the asymptotic distribution.

Variance Ratio statistic for Cointegration

Breitung (2002)

Consider a $(n \times 1)$ vector time series $\boldsymbol{Z}_t$. Suppose that there exists a full rank $(n \times n)$ matrix $\boldsymbol{Q} = (\boldsymbol{\gamma}, \boldsymbol{\beta})$, with $0 \leq h \leq n-1$, such that

$$\boldsymbol{Q}'\boldsymbol{Z}_t = \begin{bmatrix} \boldsymbol{\gamma}'\boldsymbol{Z}_t \\ \boldsymbol{\beta}'\boldsymbol{Z}_t \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{V}_{1,t} \\ \boldsymbol{V}_{2,t} \end{bmatrix}, \tag{27}$$

where $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are $[n \times (n-h)]$ and $(n \times h)$ matrices, respectively.

Variance Ratio statistic for Cointegration

Breitung (2002)

1. $\boldsymbol{V}_{1,t}$ is a $[(n-h)\times 1]$ vector of stochastic trend satisfy the functional central limit theorem, i.e.

$$T^{-1/2}\boldsymbol{V}_{1,[Tr]} \Rightarrow \boldsymbol{W}(r), \tag{28}$$

where $0 < r \leq 1$, $\boldsymbol{W}(r)$ is a $(n-h)$-dimensional standard Brownian motion.

2. $\boldsymbol{\beta}$ is well known the cointegration vectors.
3. $\boldsymbol{V}_{2,t} \equiv \boldsymbol{\beta}'\boldsymbol{Z}_t$ is a $(h\times 1)$ stationary vector.

Variance Ratio statistic for Cointegration

Breitung (2002)

The nonparametric cointegration rank test is base on the following statistic

$$\boldsymbol{R}_T = \boldsymbol{A}_T \boldsymbol{B}_T^{-1}, \tag{29}$$

where

$$\boldsymbol{A}_T = \sum_{t=1}^{T} \hat{\boldsymbol{Z}}_t \hat{\boldsymbol{Z}}_t', \; \boldsymbol{B}_T = \sum_{t=1}^{T} \hat{\boldsymbol{S}}_t \hat{\boldsymbol{S}}_t', \tag{30}$$

in which $\hat{\boldsymbol{Z}}_t = \boldsymbol{Y}_t - \hat{\boldsymbol{\alpha}}' \boldsymbol{D}_t$, $\hat{\boldsymbol{\alpha}} = (\sum_{t=1}^{T} \boldsymbol{D}_t \boldsymbol{D}_t')^{-1} \sum_{t=1}^{T} \boldsymbol{D}_t \boldsymbol{Y}_t'$, and $\hat{\boldsymbol{S}}_t = \hat{\boldsymbol{Z}}_1 + \hat{\boldsymbol{Z}}_2 + \cdots + \hat{\boldsymbol{Z}}_t$.

Variance Ratio statistic for Cointegration

Breitung (2002)

The test statistic is defined as

$$\Lambda_B = T^2 \sum_{j=1}^{n-h} \lambda_j, \tag{31}$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the ordered eigenvalues of the matrix $\boldsymbol{R}_T$.

Variance Ratio statistic for Cointegration

Breitung (2002)

Theorem (Breitung, 2002)

Under the null hypothesis that $h = h_0$ against $h > h_0$, the test statistic Λ_B has following asymptotic distribution

$$\Lambda_B \Rightarrow \operatorname{tr}\left\{ \int_0^1 \boldsymbol{W}_j(r)\boldsymbol{W}_j(r)'dr \left[\int_0^1 \tilde{\boldsymbol{W}}_j(r)\tilde{\boldsymbol{W}}_j(r)'dr \right]^{-1} \right\}, \quad (32)$$

where $\tilde{\boldsymbol{W}}_j(r) = \int_0^r \boldsymbol{W}_j(a)da$, *and*

Variance Ratio statistic for Cointegration

Breitung (2002)

$$
\begin{aligned}
\boldsymbol{W}_0(r) &\equiv \boldsymbol{W}(r), \text{ for } \boldsymbol{D}_t = 0, \\
\boldsymbol{W}_1(r) &\equiv \boldsymbol{W}(r) - \int_0^1 \boldsymbol{W}(a)da, \text{ for } \boldsymbol{D}_t = 1, \\
\boldsymbol{W}_2(r) &\equiv \boldsymbol{W}(r) - (4-6r)\int_0^1 \boldsymbol{W}(a)da - (12r-6) \\
&\quad \times \int_0^1 a\boldsymbol{W}(a)da, \text{ for } \boldsymbol{D}_t = (1,t)'.
\end{aligned}
$$

Fractional Integrated process

Definition

The fractionally integrated process for Z_t is defined as

$$Z_t = \Delta_+^{-d} u_t = \sum_{j=0}^{t-1} \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} u_{t-j}, \tag{33}$$

where $\Delta_+^{-d} = (1-L)^{-d}\mathbb{I}(t>0)$, $\mathbb{I}(\cdot)$ is the indicator function, $\Gamma(\cdot)$ is the Gemma function, and u_t is a stationary process.

Remark

When $d > 1/2$, Z_t is a nonstationary process.

Fractional Integrated process

Under some regularity conditions, a fractional functional central limit theorem can be obtained for Z_t

$$T^{1/2-d} Z_{[Tr]} \Rightarrow \sigma_Z W_d(r), \text{ for some } \sigma_Z > 0, \tag{34}$$

where $0 < r \leq 1$, $W_d(r)$ is type II fractional standard Brownian motion of order $d > 1/2$, and $W_d(r)$ is defined as

$$W_d(r) = \begin{cases} 0, \quad a.s., & r = 0, \\ \frac{1}{\Gamma(d)} \int_0^r (r-s)^{d-1} dW(s), & r > 0, \end{cases} \tag{35}$$

in which $W(r)$ is the standard Brownian motion.[2]

[2] See Marinucci and Robinson (2000).

Variance Ratio statistic

Nielson (2010)

The variance ratio statistic for the fractionally integrated process is given by

$$\rho_N(d_1) = T^{2d_1} \sum_{t=1}^{T} \hat{Z}_t^2 \left[\sum_{t=1}^{T} \hat{S}_t(d_1)^2 \right]^{-1}, \tag{36}$$

where $\hat{Z}_t = Y_t$ and $\hat{S}_t(d_1) = \Delta_+^{-d_1} \hat{Z}_t$ is the fractional partial sum.

Variance Ratio statistic

Nielson (2010)

Theorem (Nielson, 2010)

The asymptotic distribution of $\rho_N(d_1)$ *is*

$$\begin{aligned} \rho_N(d_1) &\Rightarrow \int_0^1 \sigma_Z^2 W_d(r)^2 dr \left[\int_0^1 \sigma_Z^2 W_{d+d_1}(r)^2 dr\right]^{-1} \\ &= \int_0^1 W_d(r)^2 dr \left[\int_0^1 W_{d+d_1}(r)^2 dr\right]^{-1}. \end{aligned} \tag{37}$$

Again, the nuisance parameter σ_Z is also eliminated from the asymptotic distribution.

Variance Ratio statistic

Nielson (2010)

The cointegration rank test can be extended to the fractional cointegration framework. The test is base on the following statistic

$$\boldsymbol{R}_T(d_1) = \boldsymbol{A}_T \boldsymbol{B}_T(d_1)^{-1}, \tag{38}$$

where

$$\boldsymbol{A}_T = \sum_{t=1}^{T} \hat{\boldsymbol{Z}}_t \hat{\boldsymbol{Z}}'_t,\ \boldsymbol{B}_T(d_1) = \sum_{t=1}^{T} \hat{\boldsymbol{S}}_t(d_1)\hat{\boldsymbol{S}}_t(d_1)',$$

in which $\hat{\boldsymbol{S}}_t(d_1) = \Delta_+^{-d_1} \hat{\boldsymbol{Z}}_t$ is the fractional partial sum of $\hat{\boldsymbol{Z}}_t = \boldsymbol{Y}_t - \hat{\boldsymbol{\alpha}}' \boldsymbol{D}_t$.

Variance Ratio statistic

Nielson (2010)

The test statistic is defined as

$$\Lambda_N(d_1) \;=\; T^{2d_1} \sum_{j=1}^{n-h} \lambda_j(d_1), \tag{39}$$

where $\lambda_1(d_1) \leq \lambda_2(d_1) \leq \cdots \leq \lambda_n(d_1)$ denote the ordered eigenvalues of the matrix $\boldsymbol{R}_T(d_1)$.

Variance Ratio statistic

Nielson (2010)

Theorem (Nielson, 2010)

The test statistic $\Lambda_N(d_1)$ is asymptotically distributed as a fractional functional of Brownian motions:

$$\operatorname{tr}\left\{\int_0^1 \boldsymbol{B}_{j,d}^{n-h}(r)\boldsymbol{B}_{j,d}^{n-h}(r)'dr\left[\int_0^1 \tilde{\boldsymbol{B}}_{j,d,d_1}^{n-h}(r)\tilde{\boldsymbol{B}}_{j,d,d_1}^{n-h}(r)'dr\right]^{-1}\right\}, \quad (40)$$

where the subscript j is denoted by $j=0$ when $\boldsymbol{D}_t=0$, $j=1$ when $\boldsymbol{D}_t=1$, and $j=2$ when $\boldsymbol{D}_t=(1,t)'$,

Variance Ratio statistic

Nielson (2010)

$$
\begin{aligned}
\boldsymbol{B}_{0,d}^{n-h}(r) &= \boldsymbol{W}_{d}^{n-h}(r), \\
\boldsymbol{B}_{1,d}^{n-h}(r) &= \boldsymbol{W}_{d}^{n-h}(r) - \int_0^1 \boldsymbol{W}_{d}^{n-h}(a)da, \\
\boldsymbol{B}_{2,d}^{n-h}(r) &= \boldsymbol{W}_{d}^{n-h}(r) - \int_0^1 \boldsymbol{W}_{d}^{n-h}(a)\boldsymbol{D}(a)'da \\
&\quad \times \left[\int_0^1 \boldsymbol{D}(a)\boldsymbol{D}(a)'da\right]^{-1} \boldsymbol{D}(r),
\end{aligned}
$$

Variance Ratio statistic

Nielson (2010)

$$
\begin{aligned}
\tilde{\boldsymbol{B}}^{n-h}_{0,d,d_1}(r) &= \boldsymbol{W}^{n-h}_{d+d_1}(r), \\
\tilde{\boldsymbol{B}}^{n-h}_{1,d,d_1}(r) &= \boldsymbol{W}^{n-h}_{d+d_1}(r) - \int_0^1 \boldsymbol{W}^{n-h}_{d}(a)da \int_0^r \frac{(r-a)^{d_1-1}}{\Gamma(d_1)}da, \\
\tilde{\boldsymbol{B}}^{n-h}_{2,d,d_1}(r) &= \boldsymbol{W}^{n-h}_{d+d_1}(r) - \int_0^1 \boldsymbol{W}^{n-h}_{d}(a)\boldsymbol{D}(a)'da \\
&\quad \times \left[\int_0^1 \boldsymbol{D}(a)\boldsymbol{D}(a)'da\right]^{-1} \int_0^r \frac{(r-a)^{d_1-1}}{\Gamma(d_1)}\boldsymbol{D}(a)da,
\end{aligned}
$$

in which $\boldsymbol{D}(a) = (1, a)'$ and $\boldsymbol{W}^{n-h}_{d}(r)$ is an $(n-h)\times 1$ vector representing fractional standard Brownian motion.

The Presence of Structural Breaks

1. Tests for unit roots or cointegration have low power in the presence of structural breaks.
2. Perron (1989) suggested a modification of unit root tests that allow for one break with a predetermined timing.

The Presence of Structural Breaks

Perron (1989) Test for Unit root

The model is given by:

$$Y_t = \alpha_0 + \alpha_1 D_t + \rho Y_{t-1} + u_t, \tag{41}$$

where D_t is a dummy variable defined as

$$D_t = \begin{cases} 0, & \text{if } t \leq [T\tau], \\ 1, & \text{if } t > [T\tau], \end{cases}$$

in which τ is the change point.

The Presence of Structural Breaks

Gregory and Hansen (1996) test for Cointegration

The model is given by:

$$Y_t = \alpha_0 + \alpha_1 D_t + \mathbf{x}_t'\boldsymbol{\beta} + e_t, \tag{42}$$

where D_t is a dummy variable defined as

$$D_t = \begin{cases} 0, & \text{if } t \leq [T\tau], \\ 1, & \text{if } t > [T\tau], \end{cases}$$

in which τ is the change point.

The Presence of Structural Breaks

Johansen *et al.* (2000) test for Cointegration

The model allows for any pre-specified number of sample periods (q), of lengh $T_j - T_{j-1}$ for $j = 1, \ldots, q$ and $0 = T_0 < T_1 < \cdots < T_q = T$. The model is given by:

$$\Delta \boldsymbol{Y}_t = \boldsymbol{\Pi} \boldsymbol{Y}_{t-1} + \boldsymbol{\mu}_j + \boldsymbol{\Pi}_j t + \sum_{i=1}^{k-1} \boldsymbol{\Gamma}_i \Delta \boldsymbol{Y}_{t-i} + \boldsymbol{\varepsilon}_t, \tag{43}$$

for $j = 1, \ldots, q$ and $T_{j-1} + k < t \leq T_j$

The Presence of Structural Breaks

Dechert (2012) test for Cointegration

The observed time series is allowed to have trend breaks. The model is given by:

$$\boldsymbol{Y}_t = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 t + \boldsymbol{\alpha}_2 DT_t(\tau) + \boldsymbol{Z}_t, \tag{44}$$

where

$$DT_t(\tau) = \sum_{i=1}^{t} \mathbb{I}(i > T\tau). \tag{45}$$

Advantages of the Fourier Form Approximation

1. It works reasonably well for types of breaks often used in economic analysis.
2. The Fourier function with a single frequency component can be a reasonable approximation for breaks of an unknown form even if the function itself is not periodic.
3. It involves only the determination of the appropriate frequency component in the model and hence avoids the complication of selecting break dates, the number of breaks and the form of breaks.

Singel frequency v.s. Cumulative frequencies

Becker *et al.* (2006) and Enders and Lee (2012a) illustrate that time-varying features in a univariate time series variable can be reasonably approximated by single-frequency fourier components. This form in our multivariate variables model can written as

$$Y_{i,t} = \alpha_{i,0} + \alpha_{i,1} t + \alpha_{i,2} \sin\left(\frac{2\pi k_i t}{T}\right) + \alpha_{i,3} \cos\left(\frac{2\pi k_i t}{T}\right) + Z_{i,t}, \quad (46)$$

for $i = 1, 2, \ldots, n$, where $\alpha_{i,0}$ is the intercept, $\alpha_{i,1}$ is the linear trend, $\alpha_{i,2}$ and $\alpha_{i,3}$ measure the amplitude and displacement of sinusoidal components, k_i represents a particular frequency, and $Z_{i,t}$ is the stochastic trend component of $Y_{i,t}$ in variable i.

Singel frequency v.s. Cumulative frequencies

Equation (46) is unrealistic in practice because it is hard to choose different combination of k_i in model setting.

To address the problem, it may be assumed that there is a homogeneous single-frequency component, i.e., $k_i = k$. However, this assumption may not be appropriate for multiple variables, as there could be different types of breaks between variables, which means their frequencies may differ.

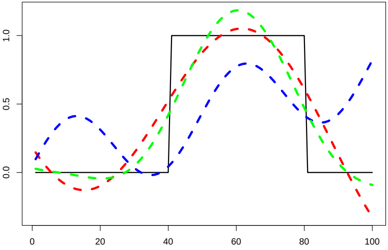
Our Model

We therefore assume the following homogeneous cumulative fourier function model to approximate the unknown forms of break in a multivariate time series model:

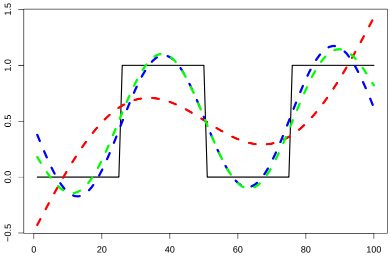
$$\begin{aligned} Y_{i,t} &= \alpha_{i,0} + \alpha_{i,1} t + \sum_{k=1}^{K} \alpha_{i,2,k} \sin\left(\frac{2\pi kt}{T}\right) \\ &\quad + \sum_{k=1}^{K} \alpha_{i,3,k} \cos\left(\frac{2\pi kt}{T}\right) + Z_{i,t}, \end{aligned} \tag{47}$$

for $i = 1, 2, \ldots, n$, where K represents the number of frequencies cumulated.

Singel frequency v.s. Cumulative frequencies



(a) temporary break



(b) multiple breaks

Figure: The red and blue dashed line represents a single frequency using $k = 1$ and $k = 2$, respectively, and the green dashed line represents the cumulative frequencies using $K = 2$.

Our Model

For convenience, we rewrite model (47) in a more compact form as follows:

$$\begin{aligned}\boldsymbol{Y}_t &= \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 t + \sum_{k=1}^{K} \boldsymbol{\alpha}_{2,k} \sin\left(\frac{2\pi kt}{T}\right) \\ &\quad + \sum_{k=1}^{K} \boldsymbol{\alpha}_{3,k} \cos\left(\frac{2\pi kt}{T}\right) + \boldsymbol{Z}_t, \end{aligned} \tag{48}$$

where $\boldsymbol{Y}_t = (Y_{1,t}, Y_{2,t}, \ldots, Y_{n,t})'$, $\boldsymbol{\alpha}_0 = (\alpha_{1,0}, \alpha_{2,0}, \ldots, \alpha_{n,0})'$, $\boldsymbol{\alpha}_1 = (\alpha_{1,1}, \alpha_{2,1}, \ldots, \alpha_{n,1})'$, $\boldsymbol{\alpha}_{2,k} = (\alpha_{1,2,k}, \alpha_{2,2,k}, \ldots, \alpha_{n,2,k})'$, and $\boldsymbol{\alpha}_{3,k} = (\alpha_{1,3,k}, \alpha_{2,3,k}, \ldots, \alpha_{n,3,k})'$.

Our Model

Given the number of frequencies K, the coefficients of the deterministic components $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_0', \boldsymbol{\alpha}_1', \boldsymbol{\alpha}_{2,1}', \ldots, \boldsymbol{\alpha}_{2,K}', \boldsymbol{\alpha}_{3,1}', \ldots,$ $\boldsymbol{\alpha}_{3,K}')'$ can be estimated using OLS,

$$\hat{\boldsymbol{\alpha}}(K) = \left[\sum_{t=1}^{T} \boldsymbol{D}_t(K)\boldsymbol{D}_t(K)'\right]^{-1} \sum_{t=1}^{T} \boldsymbol{D}_t(K)\boldsymbol{Y}_t', \tag{49}$$

where $\boldsymbol{D}_t(K) = (1, t, \sin(2\pi 1t/T), \ldots, \sin(2\pi Kt/T), \cos(2\pi 1t/T),$ $\ldots, \cos(2\pi Kt/T))'$.

Our Model

The cointegration rank test in our model is base on the following statistic

$$\boldsymbol{R}_T(d_1, K) = \boldsymbol{A}_T(K)\boldsymbol{B}_T(d_1, K)^{-1}, \tag{50}$$

where

$$\begin{aligned}
\boldsymbol{A}_T(K) &= \sum_{t=1}^{T} \hat{\boldsymbol{Z}}_t(K)\hat{\boldsymbol{Z}}_t(K)', \\
\boldsymbol{B}_T(d_1, K) &= \sum_{t=1}^{T} \hat{\boldsymbol{S}}_t(d_1, K)\hat{\boldsymbol{S}}_t(d_1, K)',
\end{aligned}$$

in which $\hat{\boldsymbol{S}}_t(d_1, K) = \Delta_+^{-d_1}\hat{\boldsymbol{Z}}_t(K)$ is the fractional partial sum of $\hat{\boldsymbol{Z}}_t(K) = \boldsymbol{Y}_t - \hat{\boldsymbol{\alpha}}(K)'\boldsymbol{D}_t(K)$.

Our Model

The nonparametric variance ratio trace statistic is defined as follows

$$\Lambda_{n,h}(d_1, K) = T^{2d_1} \sum_{j=1}^{n-h} \lambda_j(d_1, K), \tag{51}$$

where $\lambda_1(d_1, K) \leq \lambda_2(d_1, K) \leq \cdots \leq \lambda_n(d_1, K)$ denote the ordered eigenvalues of the matrix $\boldsymbol{R}_T(d_1, K)$.

Our Model

Theorem

The asymptotic distribution of $\Lambda_{n,h}(d_1, K)$ *is*

$$\Lambda_{n,h}(d_1, K) \Rightarrow U_{n-h}(d, d_1, K). \tag{52}$$

The exact form of asymptotic distribution $U_{n-h}(d, d_1, K)$ *will be completed in this project.*

Thank You

for your

Patient Listening